Univariate Taylor Approximation

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Problem 1

Your are the sales manager of a firm. Consider your firm's supply function $y = f(p)$ where p denotes the price of a commodity and y is the quantitiy of the same commodity.

Assume that the true function $y = f(p)$ is unknown, but you know, that the following points are located at the graph of the function:

1. $(p_1, y_1) = (5; 35)$
2. $(p_2, y_2) = (5.1; 36.21)$

Your boss wants you to estimate how much of the commodity your firm would supply at a price of 20.

The Taylor Theorem

Theorem

Given an arbitrary function $f(x)$, if $f(x_0)$ is known and the value of the derivatives at x_0 (i.e. $f'(x_0)$, $f''(x_0)$, etc.) are known, the function $f(x)$ can be expanded around the point x_0 as follows:

$$f(x) = \left[\frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n\right] + R_n$$

$$= \boldsymbol{F(x)} + R_n$$

To approximate an unknown function, we are interested in $F(x)$.

The Taylor Theorem

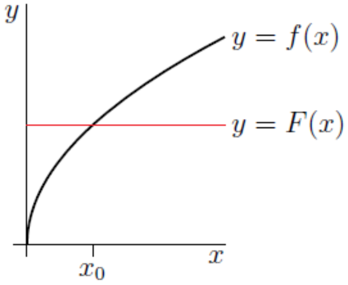
Characteristics of the Taylor Theorem

- The function $f(x)$ must be infinitely differentiable in some open interval around $x = x_0$
- The Remainder converges to zero as x approaches x_0, i.e. $\lim_{x \to x_0} R_n = 0$
- Hence, the smaller the interval, the better the approximation $F(x)$

Approximation of univariate functions

0. **Approximation of degree zero - the constant approximation**

$$F(x) = \frac{f(x_0)}{0!} = f(x_0)$$

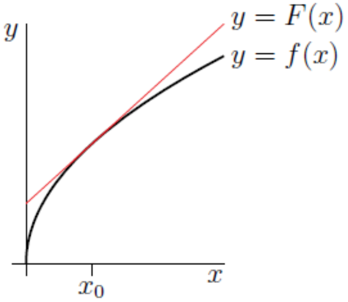


Source: Own illustration based on Feldman (2014)

Approximation of univariate functions

1. **First-degree approximation - the linear approximation**

$$F(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) = f(x_0) + f'(x_0)(x - x_0)$$



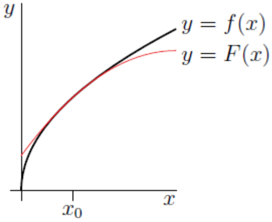
Source: Own illustration based on Feldman (2014)

Approximation of univariate functions

2. **Second-degree approximation - the quadratic approximation**

$$F(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$



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Problem 1 continued

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$$f(p) = f(p_1) + f'(p_1)(p - p_1)$$

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$$f(p_2) \overset{!}{=} y_2 = f(p_1) + f'(p_1)(p_2 - p_1)$$

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$$f(p_2) \overset{!}{=} y_2 = f(p_1) + f'(p_1)(p_2 - p_1)$$

$$36.21 = 35 + f'(p_1)(5.1 - 5)$$

Problem 1 continued

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$$f(p_2) \overset{!}{=} y_2 = f(p_1) + f'(p_1)(p_2 - p_1)$$

$$36.21 = 35 + f'(p_1)(5.1 - 5)$$

$$\Rightarrow f'(p_1) = 12.1$$

Problem 1 continued

We want to find $f(20)$! Hence, $p = 20$.

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Problem 1 continued

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$$f(p) = f(p_1) + f'(p_1)(p - p_1)$$

$$f(20) = 35 + 12.1(20 - 5)$$

$$\Rightarrow f(20) = 216.5$$

Answer: For a price of 20, your firm would supply 216.5 units of the commodity.

Problem 1 continued

In order to refine your approximation, your boss gives you another data point, that he found out.

1. $(p_1, y_1) = (5; 35)$
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Problem 1 continued

In order to refine your approximation, your boss gives you another data point, that he found out.

1. $(p_1, y_1) = (5; 35)$
2. $(p_2, y_2) = (5.1; 36.21)$
3. $(p_3, y_3) = (4.9; 33.81)$

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$$f(p) = f(p_1) + f'(p_1)(p - p_1) + f''(p_1)(p - p_1)^2$$

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$$f(p_2) \overset{!}{=} y_2 = f(p_1) + {\color{red}f'(p_1)}(p_2 - p_1) + {\color{red}f''(p_1)}(p_2 - p_1)^2$$
$$f(p_3) \overset{!}{=} y_3 = f(p_1) + {\color{red}f'(p_1)}(p_3 - p_1) + {\color{red}f''(p_1)}(p_3 - p_1)^2$$

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Hence, you can now use the quadratic approximation!

$$f(p) = f(p_1) + f'(p_1)(p - p_1) + f''(p_1)(p - p_1)^2$$

$$f(p_2) \stackrel{!}{=} y_2 = f(p_1) + f'(p_1)(p_2 - p_1) + f''(p_1)(p_2 - p_1)^2$$
$$f(p_3) \stackrel{!}{=} y_3 = f(p_1) + f'(p_1)(p_3 - p_1) + f''(p_1)(p_3 - p_1)^2$$

$$36.21 = 35 + f'(p_1)(5.1 - 5) + f''(p_1)(5.1 - 5)^2$$
$$33.81 = 35 + f'(p_1)(4.9 - 5) + f''(p_1)(4.9 - 5)^2$$

Problem 1 continued

$\Rightarrow$ $f'(p_1) = 12$

$\Rightarrow$ $f''(p_1) = 1$

We want to find $f(20)$! Hence, $p = 20$.

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Problem 1 continued

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We want to find $f(20)$! Hence, $p = 20$.

$$f(p) = f(p_1) + f'(p_1)(p - p_1) + f''(p_1)(p - p_1)^2$$

$$f(20) = 35 + 12(20 - 5) + 1(20 - 5)^2$$

Problem 1 continued

$\Rightarrow f'(p_1) = 12$

$\Rightarrow f''(p_1) = 1$

We want to find $f(20)$! Hence, $p = 20$.

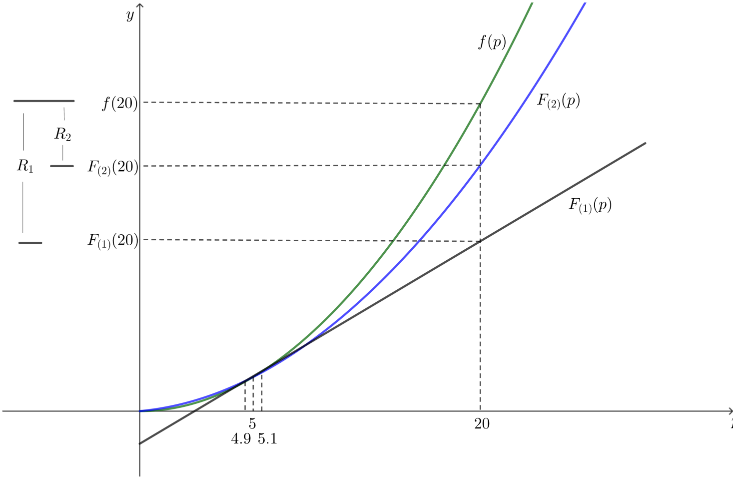
$$f(p) = f(p_1) + f'(p_1)(p - p_1) + f''(p_1)(p - p_1)^2$$

$$f(20) = 35 + 12(20 - 5) + 1(20 - 5)^2$$

$\Rightarrow f(20) = 440$

Answer: For a price of 20, your firm would supply 440 units of the commodity.

Problem 1 continued



Mathematical and Economic Applications

" [...] As it turned out, the Taylor series was of such importance that Lagrange called it "the basic principle of differential calculus." Indeed, it plays a very important part in **calculus** as well as in **computation**, **statistics**, and **econometrics**. As it is well known, a calculator or computer can only add and, in fact, can deal only with 0's and 1's. So how is it possible that you punch in a number and then press a button, and the calculator finds the logarithm or exponential of that number? Similarly, how can a machine capable of only adding **give** you **the sine and cosine of an angle**, **find solutions to an equation**, and **find the maxima and minima of a function**? All these and more can be done due to the Taylor series."

Dadkhah K. (2011) The Taylor Series and Its Applications. In: Foundations of Mathematical and Computational Economics. Springer, Berlin, Heidelberg)

Literature

- Wainwright, K. (2005) Fundamental methods of mathematical economics, pp. 242 ff., McGraw-Hill.
- Silberberg, E., Suen, W. C. (2000) The Structure of Economics: A Mathematical Analysis 3rd Edition, pp. 32 ff., McGraw-Hill.
- Feldman (2014) Taylor Polynomials - Approximating Functions Near a Specific Point, http://www.math.ubc.ca/ feldman/m105/taylor.pdf

- find this presentation on silviostaedter.weebly.com